MA948-15 Applied Scheme Theory

22/23

Department
: Warwick Mathematics Institute

Level
: Taught Postgraduate Level

Module leader
: Jose Rodrigo

Credit value
: 15

Assessment
: 100% exam

Study location
: University of Warwick main campus, Coventry

Description

Introductory description

What are schemes and what are they good for? More importantly, why should I bother?
Schemes are geometric objects patched together from (commutative, unital) rings in very much the same way that differentiable manifolds are patched from open subsets of R^n. The local models are called affine schemes, and are really the same thing as commutative rings, but from a geometric vantage point.
Whereas general schemes can be rather unwieldy and pathological, there are several important subclasses that are better behaved and amenable to deeper study (this is analogous to restricting from general topological spaces to CW complexes in topology). In particular, the algebraic varieties over algebraically closed ground fields which make up the basic objects of the introductory algebraic geometry module MA4A5 can be naturally viewed as schemes. However, and this is a first indication that schemes are useful, when one considers families of algebraic varieties, and in particular limits of varieties $\{X_t\}$ depending algebraically on a parameter t, it turns out that the limit X_0 must usually be viewed as a scheme to ensure reasonable continuity properties in the family (meaning, X_0 should under favourable conditions inherit properties from X_t). Viewing X_0 as a scheme amounts to imagining it being endowed with some “infinitesimal fluff” that remembers the limiting procedure in a certain sense. This can be made precise using nilpotent elements in the structure sheaves. Thus this is a first area (entirely within classical algebraic geometry, over the complex numbers, say) where schemes provide an advantageous theoretical set-up: the theory of deformations (and degenerations) of algebraic varieties. Studying properties of varieties via degeneration goes way back to at least the classical Italian school of algebraic geometry, but has recently reached new levels of sophistication and seen a boost of

activity.
Another main area where schemes are useful is number theory: classical algebraic varieties are essentially solution sets, in some fixed algebraically closed ground field, of a bunch of polynomial equations (in many variables). But polynomial equations make sense over any ring R, and then you may be interested in solutions in R, or any overring S of R. For example, R may be the integers. You can then also reduce the defining polynomial equations modulo various primes, and try to relate the properties of the solutions of the reductions to that of the initial equations (essentially, the idea of degeneration again). Schemes provide a way to talk about these processes in a geometric way. In particular, they allow to establish and pursue useful analogies, for example between the rings of integers in an algebraic number field and algebraic curves. It can be argued that schemes do not quite capture everything number theorists care for (they tend to miss “what is going on at Archimedean primes”, Andr\'{e} Weil’s criticism of scheme theory), but schemes go a long way towards a unified perspective on number theory and geometry (“Kronecker’s Jugendtraum”).

The “applied” in the title of the Module refers to the fact that we will be interested throughout in applications of schemes to problems in other areas that can be formulated without recurse to scheme language, but for which scheme theory supplies valuable tools. This is similar to the meaning of the qualifier in “Calculus for the practical person”! We are not interested in learning tons of definitions in the arid generality of general scheme theory and only know the names of things, we want to understand how things work and solve problems!

Module aims

This Module seeks to convey to the students the usefulness of scheme theory for solving problems in other areas, such as number theory or deformation-theoretic questions in classical algebraic geometry. It also seeks to provide them with the tools to use scheme theory in easy new contexts and to explore some of these issues further on their own.

Outline syllabus

This is an indicative module outline only to give an indication of the sort of topics that may be covered. Actual sessions held may differ.

-Schemes and sheaves: definitions and first examples
-Basic attributes: reduced, irreducible, finite type; separated schemes, proper morphisms
-Fibre products; uses of generic points
-Proj and blow-ups; some invariants of projective schemes: Hilbert function and Hilbert polynomial
-Flat limits and basics about deformations and degenerations
-Schemes in arithmetic: ground fields, base rings; connections to Galois theory; the Frobenius morphism
-Group schemes; the functor of points
-Kaehler differentials and smooth morphisms

Learning outcomes

By the end of the module, students should be able to:

- -Thorough familiarity with the fundamental notions of scheme theory -Geometric intuition for properties peculiar to schemes such as non-reducedness -A good command of some of the fundamental tools added by scheme-theory to the stock of classical algebraic geometry -the potential for solving easy unseen problems using a scheme-theoretic point of view

Indicative reading list

D. Mumford, T. Oda: Algebraic Geometry II, Texts and Readings in Mathematics 73, Hindustan Book Agency (2015)

Y. Manin: Introduction to the Theory of Schemes, Moscow Lectures 1, Springer (2018)

D. Eisenbud, J. Harris: The Geometry of Schemes, Graduate Texts in Math. 197, Springer (2000)

Subject specific skills

-Thorough familiarity with the fundamental notions of scheme theory
-Geometric intuition for properties peculiar to schemes such as non-reducedness
-A good command of some of the fundamental tools added by scheme-theory to the stock of classical algebraic geometry
-the potential for solving easy unseen problems using a scheme-theoretic point of view

Transferable skills

- sourcing research material
- prioritising and summarising relevant information
- absorbing and organizing information
- presentation skills (both oral and written)

Study

Study time

Type	Required
Lectures	30 sessions of 1 hour (100%)
Total	30 hours

Private study description

Review lectured material.
Work on suplementary reading material.
Source, organise and prioritise material for additional reading.

Costs

No further costs have been identified for this module.

Assessment

You must pass all assessment components to pass the module.

Assessment group A

	Weighting	Study time
Oral Exam	100%	
An oral exam involving a presentation by the student, followed by questions from the panel (2 members of the department)		

Feedback on assessment

Students will receive feedback from the course instructor after the oral exam, to cover also areas like presentation skills and use of technologies (or blackboard)

Past exam papers for MA948

Availability

There is currently no information about the courses for which this module is core or optional.